

Issues concerning the waterfall of hybrid inflation

David H. LYTH

Department of Physics, Lancaster University, Lancaster LA1 4YB, UK

We discuss the waterfall that ends hybrid inflation. Making some simplifying assumptions, that may be satisfied by GUT inflation models, two issues are addressed. First, the procedure of keeping the quantum fluctuation of the waterfall field only in the regime where it can be regarded as classical. Second, the contribution to the primordial curvature perturbation that is generated during the waterfall. Because the waterfall field is heavy during inflation, the spectrum of this contribution is strongly blue and hence is negligible on cosmological scales.

§1. Introduction

The initial condition for the observable universe presumably is set by an early era of inflation.¹⁾ To generate the observed primordial curvature perturbation, inflation should be almost exponential while cosmological scales leave the horizon, corresponding to an almost constant Hubble parameter $H(t)$. The simplest way of achieving that is to invoke slow-roll inflation, where $H(\phi(t))$ depends only on a slowly varying inflaton field ϕ .

The particular type of slow-roll inflation called hybrid inflation invokes, in addition to the inflaton, a waterfall field χ . The inflaton has zero vev while the waterfall has a nonzero vev. Until inflation nears its end, χ is fixed at the origin by a positive mass-squared $m^2(\phi(t))$, up to a vacuum fluctuation that is dropped. The displacement of χ from its vev is supposed to generate most of the inflationary potential.

When $\phi(t)$ falls through some critical value, $m^2(\phi)$ goes negative. Then, for wavenumbers below some maximum k_{\max} , the vacuum fluctuation $\chi_{\mathbf{k}}(t)$ becomes classical. Only the classical modes are kept. The spatially averaged field $\chi(t)$ grows with time, and eventually becomes equal to the vev of χ . This growth of χ is called the waterfall because it is supposed to happen quickly.

In this article we consider the waterfall era without reference to a particular inflationary potential, under some simplifying assumptions. Two fundamental issues present themselves.

One of them is the dropping of the vacuum fluctuation in the quantum regime. The motivation for this comes from the well-known fact that it gives infinite contributions to the mean-square field, the energy density and the pressure. With a cutoff at momentum Λ_{UV} , the energy density is $\rho_{\text{vluc}} = \Lambda_{\text{UV}}^4/16\pi^2$, which is widely regarded as a contribution to the cosmological constant. In fact, that is not a viable interpretation, because the vacuum fluctuation contributions gives pressure $P_{\text{vluc}} = \Lambda_{\text{UV}}^4/48\pi^2$, whereas the cosmological constant has $P_\Lambda = -\rho_\Lambda$. We will discuss some alternative proposals to deal with the vacuum fluctuation, but none are relevant for the waterfall and we stay with the procedure dropping the quantum regime.

Our second issue concerns the contribution ζ_χ to the primordial curvature per-

turbation ζ , that is generated during the waterfall. In accordance with the received wisdom for a field that is heavy before the waterfall begins, we find that the spectrum of this contribution goes like k^3 . On the horizon scale at the end of inflation, it might be big enough for significant black hole formation which would constrain the parameters of the hybrid inflation model, but on cosmological scales it will almost certainly be negligible.

The layout of the paper is as follows. In Section 2 we review hybrid inflation. In Section 3 we state our assumptions and identify the part of parameter space in which they are valid. In Section 4 we consider the mode decomposition of the waterfall field and in Section 5 the evolution of the classical field. In Section 6 we calculate the contribution of the waterfall to the curvature perturbation. In Section ?? we show how our results can apply to GUT inflation, and we conclude in Section 7.

§2. Hybrid inflation

We are concerned only with the simplest kind of slow-roll inflation, which assumes Einstein gravity and an inflaton with the canonical kinetic term. The energy density is $\rho = 3M_{\text{P}}^2 H^2$, where $M_{\text{P}} = (8\pi G)^{-1/2} = 2 \times 10^{18}$ GeV is the reduced Planck scale. During slow-roll inflation the following are good approximations:

$$\rho = 3M_{\text{P}}^2 H^2 \simeq V(\phi), \quad |\dot{H}| \ll H^2, \quad 3H\dot{\phi} \simeq -V'(\phi), \quad |\ddot{\phi}| \ll H|\dot{\phi}|. \quad (2.1)$$

(Throughout this paper, an over-dot means differentiation with respect to t while a prime means differentiation with respect to the displayed argument.) On scales leaving the horizon during slow-roll inflation, the perturbation $\delta\phi$ gives a time-independent contribution to the curvature perturbation, whose spectrum is

$$\mathcal{P}_{\zeta_\phi}(k) = \left(H^2 / 2\pi\dot{\phi} \right)^2, \quad (2.2)$$

with the right hand side evaluated at the epoch of horizon crossing $k = aH$.

Hybrid inflation is a particular kind of slow-roll inflation. It was proposed²⁾ as a way of solving a worry about axion cosmology,³⁾ that subsequently was more or less laid to rest.⁴⁾ Hybrid inflation was soon found^{5)-7),9)} to be a powerful tool for model building especially in the context of supersymmetry.^{#1}

We will adopt the potential

$$V(\phi, \chi) = (V_0 + V(\phi)) + \frac{1}{2} (-m^2 + g^2\phi^2) \chi^2 + \frac{1}{4} \lambda \chi^4,$$

with $0 < \lambda \ll 1$ and $g \ll 1$. The second bracket is $m^2(\phi)$, which goes negative when ϕ falls below m/g . During inflation χ is taken to vanish so that V is given by the first bracket. The vev of ϕ vanishes and we take $V(\phi)$ to vanish at the vev. The inflationary potential is supposed to be dominated by V_0 which means $V(\phi) \ll V_0$.

^{#1} More recently, there is interest in hybrid inflation with a non-canonical kinetic term, in particular DBI inflation.¹⁰⁾ The only essential feature, needed to make sense of the hybrid inflation paradigm, is that the potential dominates the energy density.

The requirements that V and $\partial V/\partial\chi$ vanish in the vacuum give the vev χ_0 and the inflation scale V_0 .

$$\chi_0^2 = m^2/\lambda, \quad V_0 = m^4/2\lambda, \quad (2.3)$$

leading to

$$6(H/m)^2 = (\chi_0/M_P)^2. \quad (2.4)$$

Minor variants of Eq. (2.3) would make little difference to our analysis. The interaction $g^2\phi^2\chi^2$ might be absent or suppressed, so that ϕ^2 is replaced by a higher power. The term $\lambda\chi^4$ might be absent or suppressed, to be replaced by a higher power. For our purpose, these variants are equivalent to allowing (respectively) g and λ to be many orders of magnitude below unity. When making estimates, we will assume instead that these parameters are of order 10^{-1} to 10^{-2} . Also, ϕ might have two or more components that vary during inflation. Our analysis will apply to that case, if at each instant the field basis is chosen so that only ϕ varies.

More drastic modifications are also possible, including inverted hybrid inflation¹¹⁾ where ϕ is increasing during inflation, and mutated/smooth hybrid inflation¹²⁾ where the waterfall field varies during inflation. Our analysis does not apply to those cases.

Taking $\chi(t)$ to be unperturbed during the waterfall, the field equations are

$$\ddot{\phi} + 3H\dot{\phi} + \nabla^2\phi = -\partial V/\partial\phi = -V'(\phi) - g^2\chi^2\phi \quad (2.5)$$

$$\ddot{\chi} + 3H\dot{\chi} + \nabla^2\chi = -\partial V/\partial\chi = -(-m^2 + g^2\phi^2)\chi, \quad (2.6)$$

and the spatially-averaged energy density and pressure are

$$\rho(t) \equiv \langle\rho\rangle = \langle V(\phi, \chi) \rangle + \langle \dot{\phi}^2 \rangle + \langle \dot{\chi}^2 \rangle \quad (2.7)$$

$$P(t) \equiv \langle P \rangle = -\langle V(\phi, \chi) \rangle + \langle \dot{\phi}^2 \rangle + \langle \dot{\chi}^2 \rangle. \quad (2.8)$$

By virtue of the field equations the energy continuity equation is satisfied:

$$\dot{\rho}(t) = -3H(t)(\rho(t) + P(t)). \quad (2.9)$$

Eqs. (2.5)–(2.8) hold if ϕ and χ are real fields (with canonical kinetic terms). In realistic models they are at least the moduli of complex fields. More generally they correspond to directions in a field space that provides a representation of some non-Abelian symmetry group (the GUT symmetry, for the waterfall field of GUT inflation). That introduces some trivial modifications, namely numerical factors in front of Eqs. (2.7) and (2.8), and further factors when it comes to the generation of χ from the vacuum fluctuation. For the waterfall field it also introduces the non-trivial and essential fact that domain walls will not form when the waterfall spontaneously breaks the symmetry $\chi \rightarrow -\chi$, which would be fatal to the cosmology. Instead there will be at most cosmic strings, which are harmless if the inflation scale is not too high. For clarity we pretend that ϕ and χ are real fields.

The potential (2.3) was proposed in,²⁾ with $V(\phi) = m_\phi^2\phi^2/2$. But if one demands that the curvature perturbation on cosmological scales is dominated by ζ_ϕ , observation requires $V''(\phi) < 0$ while those scales leave the horizon. Many forms of $V(\phi)$ have been proposed which satisfy that requirement,^{1),13)} and we will not assume any particular form.

§3. Generating the tachyonic mass

To get firm estimates, we make the following assumptions; (i) the waterfall starts during slow-roll inflation,^{#2} (ii) ϕ and $\dot{\phi}$ have negligible variation during the waterfall, (iii) the waterfall takes much less than a Hubble time.

Since $H \equiv \dot{a}/a$ changes little in a Hubble time, we can take it to have a constant value denoted simply by H . Except insofar as it generates H , we also ignore the variation of a during the waterfall, setting it equal to 1. We set $t = 0$ at the beginning of the waterfall and use a dimensionless time $\tau \equiv \mu t$. Then we write

$$m^2(\phi(t)) = -\mu^3 t, \quad \mu^3 \equiv -2g^2 \phi \dot{\phi} \simeq -2gm\dot{\phi}, \quad (3.1)$$

with μ taken to be constant.

To estimate $\dot{\phi}$ we use Eq. (2.2), evaluated for the wavenumber $k_{\text{end}} = H$ that corresponds to the horizon at the end of inflation. On cosmological scales, observation gives for the total curvature perturbation $\mathcal{P}_\zeta^{1/2}(k) = 5 \times 10^{-5}$. Let us define a number f by

$$5 \times 10^{-5} f \equiv \mathcal{P}_{\zeta_\phi}^{1/2}(k_{\text{end}}). \quad (3.2)$$

Inflation models are typically constructed so that ζ_ϕ accounts for ζ on cosmological scales.^{1),13)} They also typically make $\mathcal{P}_{\zeta_\phi}(k)$ almost scale-independent and then $f = 1$, but it's possible to choose the potential so that $\mathcal{P}_{\zeta_\phi}(k)$ is strongly increasing.¹⁴⁾ In the latter case one can have $\mathcal{P}_{\zeta_\phi}^{1/2}(k_{\text{end}}) \sim 10^{-1}$ corresponding to $f \sim 10^3$. As mentioned in the Introduction, that will give significant black hole contribution and constrain the parameters of the potential. Finally, there is the possibility that ζ_ϕ is negligible, ζ being generated after inflation by a curvaton-type mechanism;¹⁵⁾ then $f \ll 1$.

Using Eqs. (2.2) and (3.1) we have

$$\left(\frac{\mu}{m}\right)^3 \simeq 3 \times 10^3 \frac{g}{f} \left(\frac{H}{m}\right)^2. \quad (3.3)$$

Our assumption $Ht_{\text{vev}} \ll 1$ corresponds to

$$Ht_{\text{nl}} \equiv (H/\mu)\tau_{\text{nl}} \ll 1. \quad (3.4)$$

What about the consistency of the assumption that μ is constant? From Eq. (3.1), we see that this is equivalent to $\dot{\phi}$ and ϕ both having negligible variation (barring an unlikely cancellation). Let us assume for the moment that the last term of Eq. (2.5) remains negligible throughout the waterfall, so that slow-roll inflation (Eq. (2.1)) continues to hold. Then ϕ has little change during the waterfall (because it has little change in Hubble time). Negligible variation for ϕ means $|\dot{\phi}|t_{\text{nl}} \ll m/g$ which is equivalent to

$$(\mu/m)^2 \ll \tau_{\text{nl}}^{-1}. \quad (3.5)$$

^{#2} The alternative is for slow roll to end at $\phi > m/g$. Then ϕ will start to oscillate about its minimum with an amplitude that is (at least) Hubble-damped. In that case inflation continues until the amplitude of the oscillation falls below m/g .

Now consider the last term of Eq. (2.5). As we shall see, χ^2 is proportional to a prefactor times $\exp(\frac{4}{3}\tau^{3/2})$. If we insert a constant ϕ into the last term of Eq. (2.5) this gives a contribution $\Delta\dot{\phi} \sim -g^2\phi\chi^2/\mu$. Such a contribution has to be negligible if $\dot{\phi}$ is to remain slowly varying, and requiring that we find

$$g^2 \ll \lambda(\mu/m)^4. \quad (3.6)$$

We shall see that the condition for a classical regime to exist is $\tau_{\text{nl}} \gg 1$, which means that Eqs. (3.5) and (3.4) require the hierarchies

$$H \ll \mu \ll m, \quad g^2 \ll \lambda. \quad (3.7)$$

§4. Waterfall field

4.1. Quantum and classical contributions

Working with the comoving coordinate \mathbf{x} and Fourier components

$$\chi_{\mathbf{k}}(t) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \chi(\mathbf{x}, t). \quad (4.1)$$

The field equation (2.5) is

$$\ddot{\chi}_{\mathbf{k}} + 3H\dot{\chi}_{\mathbf{k}} + [-\mu^3 t + (k/a)^2] \chi_{\mathbf{k}} = 0. \quad (4.2)$$

Since we are assuming that the waterfall takes much less than a Hubble time, we can set $H = 0$ in this equation. Going to the dimensionless time $\tau \equiv \mu t$ it becomes

$$\frac{d^2 \chi_{\mathbf{k}}(\tau)}{d\tau^2} = x(\tau, k) \chi_{\mathbf{k}}(\tau), \quad x \equiv \tau - k^2/\mu^2. \quad (4.3)$$

The solutions of Eq. (4.3) are the Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$.

In the quantum theory, $\chi_{\mathbf{k}}$ becomes an operator $\hat{\chi}_{\mathbf{k}}$. Working in the Heisenberg picture we write

$$\hat{\chi}_{\mathbf{k}}(\tau) = \chi_k(\tau) \hat{a}_{\mathbf{k}} + \chi_k^*(\tau) \hat{a}_{-\mathbf{k}} \quad (4.4)$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{p}}] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{p}). \quad (4.5)$$

The mode function χ_k satisfies Eq. (4.3) and its Wronskian is normalized to -1 . We choose

$$\chi_k = \sqrt{\pi/2\mu} [\text{Bi}(x) + i\text{Ai}(x)]. \quad (4.6)$$

To understand this choice we assume that our approximations work back to a time $\tau_{\text{initial}} \ll -1$. Then there is for all k a regime $x \ll -1$, in which

$$\chi_k(\tau) = (2\mu)^{-1/2} |x|^{-1/4} e^{-i\pi/4} e^{-\frac{2}{3}i|x|^{3/2}}. \quad (4.7)$$

In this regime, x is slowly varying ($|dx/d\tau| \ll |x|$), and $\chi_{\mathbf{k}}$ describes particles with momentum \mathbf{k} and mass-squared $m^2(\phi(t))$. We choose the state vector as the vacuum,

such that $\hat{a}_{\mathbf{k}}|>=0|>$. A significant occupation number is excluded since the resulting positive pressure would spoil inflation.¹⁾#3

Using the mode function (4.6) we can work out the two-point correlator of $\hat{\chi}_{\mathbf{k}}$:

$$\langle \hat{\chi}_{\mathbf{k}}(\tau) \hat{\chi}_{\mathbf{p}}(\tau) \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{p}) \mathcal{P}_{\chi}(k, \tau) \quad (4.8)$$

$$\mathcal{P}_{\chi}(k, \tau) \equiv \frac{k^3}{2\pi^2} |\chi_k(\tau)|^2. \quad (4.9)$$

The spectrum \mathcal{P}_{χ} is independent of the direction of \mathbf{k} because the vacuum is invariant under rotations. This gives the expectation value of $\hat{\chi}^2$

$$\langle \hat{\chi}^2(\tau) \rangle = \int_0^{\infty} \frac{dk}{k} \mathcal{P}_{\chi}(k, \tau), \quad (4.10)$$

and the expectation values of $\hat{\rho}_{\chi}$ and \hat{P}_{χ} ,

$$\langle \hat{\rho}_{\chi}(t) \rangle = \frac{1}{4\pi^2} \int_0^{\infty} dk k^2 [-\mu^2 \tau |\chi_k|^2 + |\dot{\chi}_k|^2 + k^2 |\chi_k|^2] \quad (4.11)$$

$$\langle \hat{P}_{\chi}(t) \rangle = \frac{1}{4\pi^2} \int_0^{\infty} dk k^2 \left[\mu^2 \tau |\chi_k|^2 + |\dot{\chi}_k|^2 - \frac{1}{3} k^2 |\chi_k|^2 \right]. \quad (4.12)$$

They are independent of \mathbf{x} because the vacuum is invariant under translations.

We are going to see that the final value τ_{nl} is much bigger than 1. For any $\tau \gg 1$, there exists a regime $x \equiv \tau - (k/\mu)^2 \gg 1$ in which

$$\chi_k(\tau) \simeq (2\mu)^{-1/2} x^{-1/4} e^{\frac{2}{3}x^{3/2}}, \quad \dot{\chi}_k \simeq \mu \sqrt{x} \chi_k, \quad (4.13)$$

the errors vanishing in the limit $x \rightarrow \infty$. Since the phase of χ_k is now constant, $\hat{\chi}_{\mathbf{k}}(\tau) = \chi_k(\tau)(\hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}})$ to high accuracy. As a result, $\hat{\chi}_{\mathbf{k}}$ is a constant operator times a c-number, which means that $\chi_{\mathbf{k}}$ is a classical quantity in the WKB sense. By this, we mean that a measurement of $\chi_{\mathbf{k}}$ at a given time will give a state that corresponds to a definite value $\chi_{\mathbf{k}}$ at all future times.¹⁾

After such a measurement, $\chi_{\mathbf{k}}$ is a classical field, and the vev $\langle \rangle$ refers to the ensemble of universes corresponding to different outcomes of the measurement. (We have nothing to say about the cosmic Schrodinger's Cat problem that now presents itself.) The classical perturbation $\chi(\mathbf{x}, \tau)$, built from the classical modes, is statistically homogeneous and isotropic. Its mean-square is given by Eq. (4.10), keeping only the classical modes. Up to cosmic variance, the vev in Eq. (4.8) can be regarded as an average over a cell d^3k in our universe, and the vev in Eq. (4.10) can be regarded as a spatial average in our universe.

4.2. Dropping the quantum contribution

Before continuing, we comment on the procedure of dropping the quantum regime. We saw earlier that $\chi_{\mathbf{k}}$ has a particle interpretation in the regime $x \ll -1$. Consider the part of this regime in which the particles have negligible mass, $|m^2(\phi)| \ll k^2$. The vacuum fluctuation of a massless free scalar field gives

^{#3} There is still the problem with the vacuum state, that we discuss later.

infinite contributions to the mean-square field, the energy density and the pressure. The usual procedure for avoiding the infinity is to drop the vacuum fluctuation. As we now explain, that raises some questions especially when the quantum regime is accompanied by a classical regime as in the present case. Cutting out the contributions above some momentum Λ_{UV} , Eqs. (4.11) and (4.12) give in the limit of large Λ_{UV} the remaining contributions:

$$\chi_{\text{vfluc}}^2 = \Lambda_{\text{UV}}^2 / 6\pi^2 \quad (4.14)$$

$$\rho_{\text{vfluc}} = 3P_{\text{vfluc}} = \Lambda_{\text{UV}}^4 / 16\pi^2. \quad (4.15)$$

Following,¹⁶⁾ this constant energy density is widely regarded as a contribution to the cosmological constant. That is not a viable interpretation though¹⁷⁾ because the cosmological constant is Lorentz-invariant, hence of the form $T_{\mu\nu}^A = -\eta_{\mu\nu}\rho_A$. This makes $P_A = -\rho_A$ in contradiction with Eq. (4.15).

The violation of Lorentz invariance by the vacuum fluctuation may seem surprising, since the vacuum is Lorentz invariant. The violation comes of course from the momentum cutoff, which breaks Lorentz invariance.

In this discussion we have been setting $a = 1$. To understand the relation $\rho_{\text{vfluc}} = 3P_{\text{vfluc}}$ we need to restore $a(t)$, so that Eq. (4.15) becomes¹⁸⁾

$$\rho_{\text{vfluc}} = 3P_{\text{vfluc}} = \frac{1}{16\pi^2} \left(\frac{\Lambda_{\text{UV}}}{a(t)} \right)^4. \quad (4.16)$$

With Λ_{UV} time-independent, this satisfies the energy continuity equation (2.9). In fact, each k mode satisfies the energy continuity equation, because its action is invariant under the conformal time translation $d\eta = dt/a$. Imposing instead a time-independent physical cutoff $\Lambda_{\text{UV}}^{\text{physical}} = a(t)\Lambda_{\text{UV}}(t)$ makes ρ_{vac} and P_{vac} time-independent. The energy continuity equation is now violated, because the physical cutoff breaks the invariance of the action under the conformal time translation.

Instead of a momentum cutoff one might invoke a regularisation that preserves the Lorentz invariance, like Pauli-Villars¹⁹⁾ or dimensional regularization.²⁰⁾ The latter gives $\rho_{\text{vfluc}} = P_{\text{vfluc}} = 0$ for the massless case, and for mass m it gives

$$\rho_{\text{vfluc}} = -P_{\text{vfluc}} = -\frac{m^4}{64\pi^2} \left[\ln \left(\frac{\Lambda_{\text{UV}}^2}{m^2} \right) + \frac{3}{2} \right]. \quad (4.17)$$

It is not clear what that implies for the waterfall field, whose mass-squared is negative (tachyonic) during the waterfall.

We are going to keep only the classical regime $x \gg 1$, which means that we drop not just the massless regime but the entire regime $x \lesssim 1$. The procedure of keeping only the classical regime is the usual one in cosmology.¹⁾ It is applied not only to the waterfall field, but also to scalar fields that may be created after inflation by preheating mechanisms, and to light scalar fields during inflation. In this last case though, alternative procedures have been advocated^{18), 21)–23)} that we should mention here.

The light fields during inflation (taken to be almost exponential) are essentially defined as those that have the canonical kinetic term, and are practically free and

massless in all modes. (To be more precise, their potential is practically quadratic, the effective masses-squared of their perturbations are much less than H^2 in magnitude, and their rate of change is given by the slow-roll approximation $\dot{\phi} = -V'/3H$.) A canonically-normalized inflaton is one example, and there may be others including a curvaton-type field that generates a scale-independent contribution to ζ after inflation. For each mode, there is a quantum regime $k \gtrsim aH$, which for sufficiently small k gives way to a classical regime $k \ll aH$. The transition between the two regimes is roughly the epoch of horizon exit $k = aH$. The usual procedure is to keep all of the classical modes and to drop all of the rest.

The alternative procedures make no distinction between quantum and classical modes. They drop part of the classical contribution to the energy density, while keeping part of the quantum contribution. This invalidates the standard result that a light field perturbation is gaussian with spectrum (soon after horizon) $(H/2\pi)^2$. The resulting modification of the standard formula $\mathcal{P}_{\zeta_\phi}(k) = (H^2/2\pi\dot{\phi})^2$ has been worked out for one proposal.^{(21), (22)}

§5. Classical waterfall field

5.1. Spatially averaged quantities

We are dropping modes with $k > k_{\max}(\tau)$, where the cutoff satisfies

$$x(k_{\max}) \equiv \tau - (k_{\max}/\mu)^2 \gg 1. \quad (5.1)$$

The exponential amplification of $\chi(\tau)$ in the classical regime will make the results insensitive to the precise choice of the cutoff.

The spectrum of the classical field vanishes for $k > k_{\max}$, and for smaller k is given by

$$\mathcal{P}_\chi(k, \tau) = \frac{k^3}{4\pi^2\mu} x^{-1/2} e^{\frac{4}{3}x^{3/2}}, \quad (k < k_{\max}(\tau)). \quad (5.2)$$

At fixed τ , the maximum of $\mathcal{P}_\chi(k)/k$ is at k_{peak} given by

$$(k_{\text{peak}}/\mu)^2 \simeq \frac{1}{2}\tau^{-1/2} \ll 1. \quad (5.3)$$

Modes with $k \gtrsim \mu$ are negligible which means that the quantum to classical transition is practically complete when τ first becomes much bigger than 1, ie. as soon as the classical era begins.

The mean-square waterfall field is

$$\chi^2(\tau) \equiv \langle \chi^2(\tau) \rangle \simeq \frac{1}{4\pi^2\mu} \int_0^\mu x^{-1/2} e^{\frac{4}{3}x^{3/2}} k^2 dk. \quad (5.4)$$

Using $w \equiv (2/3)\sqrt{\tau}(\tau - x)$ as the integration variable, with the identity

$$\int_0^\infty dw \sqrt{w} e^{-w} = \sqrt{\pi}, \quad (5.5)$$

we find

$$\chi^2(\tau) \simeq \frac{\sqrt{\pi/2}}{16\pi^2} \mu^2 \tau^{-5/4} e^{\frac{4}{3}\tau^{3/2}}, \quad (5.6)$$

the error vanishing in the limit $\tau \rightarrow \infty$.

Since the dominant modes of χ have $k \ll \mu$, its spatial gradient is negligible (compared with its time derivative). Ignoring the spatial gradient terms in Eqs. (4.11) and (4.12) we have

$$\rho_\chi(\tau) \simeq -\frac{1}{2}\mu^2\tau\chi^2(\tau) + \frac{1}{2}\dot{\chi}^2(\tau) \simeq 0 \quad (5.7)$$

$$P_\chi(\tau) \simeq \frac{1}{2}\mu^2\tau\chi^2(\tau) + \frac{1}{2}\dot{\chi}^2(\tau) \simeq \mu^2\tau\chi^2(\tau) \simeq \dot{\chi}^2(\tau). \quad (5.8)$$

When the waterfall ends, $P_\chi(\tau) \sim \lambda^{-1}\mu^2m^2$. This is much less than the total $P \simeq -V_0 \simeq \lambda^{-1}m^4$, as is required for the consistency of our assumptions.

From Eq. (2.4) the waterfall ends at

$$\tau_{\text{nl}} \sim (\ln(\chi_0/\mu))^{2/3} \sim \left(\ln(m/\lambda^{1/2}\mu)\right)^{2/3} \gg 1. \quad (5.9)$$

To get an upper bound on τ_{nl} we use $3M_{\text{P}}^2H^2 \simeq V_0 = \chi_0^2m^2/2$ and $H \ll \mu \ll m$, which give

$$\tau_{\text{nl}} \lesssim (\ln(M_{\text{P}}/H))^{2/3}. \quad (5.10)$$

The inflationary Hubble parameter H has an upper bound coming from the absence of an observed tensor perturbation: $H \lesssim 10^{-4}M_{\text{P}}$, which if saturated would give $\tau_{\text{nl}} \lesssim 6$. In most inflation models H is not many orders of magnitude smaller (though small enough that the tensor perturbation will never be observed). In principle though, the only bound is $\sqrt{M_{\text{P}}H} > \text{MeV}$ (to allow successfully Big Bang Nucleosynthesis). This corresponds to $H > 10^{-42}M_{\text{P}}$ or $\ln(M_{\text{P}}/H) \lesssim 90$. We conclude that τ_{ev} is probably $\lesssim 10$ and certainly less than 60 or so.

5.2. Pressure perturbation

Since the spatial gradient of χ is negligible, Eqs. (5.7) and (5.8) hold locally and we have

$$\delta P_\chi(\mathbf{x}, \tau) = \mu^2\tau\delta[\chi^2(\mathbf{x}, t)]. \quad (5.11)$$

The spectrum of the pressure perturbation is

$$\mathcal{P}_{\delta P_\chi} \simeq \mu^4\tau^2\mathcal{P}_{\chi^2}. \quad (5.12)$$

Let us estimate \mathcal{P}_{χ^2} .

According to our approximations, there is no correlation between $\chi_{\mathbf{k}}$ except for the two-point correlator Eq. (4.8) and the disconnected 2^n point correlators starting with

$$\langle\chi_{\mathbf{k}_1}\chi_{\mathbf{k}_2}\chi_{\mathbf{k}_3}\chi_{\mathbf{k}_4}\rangle = \langle\chi_{\mathbf{k}_1}\chi_{\mathbf{k}_2}\rangle\langle\chi_{\mathbf{k}_3}\chi_{\mathbf{k}_4}\rangle + \text{permutations}. \quad (5.13)$$

This allows us to calculate the spectrum $\mathcal{P}_{\delta P_\chi}(k)$, using the convolution theorem,

$$(\chi^2)_{\mathbf{k}} = \frac{1}{(2\pi)^3} \int d^3k' \chi_{\mathbf{k}'} \chi_{\mathbf{k}-\mathbf{k}'}. \quad (5.14)$$

One finds²⁴⁾

$$\mathcal{P}_{\chi^2}(k, \tau) = \frac{k^3}{2\pi} \int d^3k' \frac{\mathcal{P}_\chi(k', \tau)}{k'^3} \frac{\mathcal{P}_\chi(|\mathbf{k} - \mathbf{k}'|, \tau)}{|\mathbf{k} - \mathbf{k}'|^3}. \quad (5.15)$$

We shall use this expression to estimate the contribution to the curvature perturbation that is generated during the waterfall. For that purpose we need only $k \ll H$ while the dominant modes of χ have $k \gg H$. We can therefore set $|\mathbf{k}' - \mathbf{k}|$ equal to k' on the right hand side of Eq. (5.15), to get

$$\mathcal{P}_{\chi^2}(k, \tau) = \frac{k^3}{8\pi^4 \mu^2} \int_0^{k_{\max}(\tau)} dk k^2 x^{-1} e^{\frac{8}{3}x^{3/2}}. \quad (5.16)$$

(We are using k also as the integration variable, so that the definition $x \equiv \tau - (k/\mu)^2$ continues to hold.) Proceeding as for Eq. (5.4) we find in the limit $\tau \rightarrow \infty$

$$\mathcal{P}_{\chi^2}(k, \tau) \simeq \frac{\sqrt{\pi}}{2^7 \pi^4} k^3 \mu \tau^{-7/4} e^{\frac{8}{3}\tau^{3/2}}. \quad (5.17)$$

After smoothing on a scale k , the mean-square perturbation in χ^2 is of order $\mathcal{P}_{\chi^2}(k)$, and the fractional mean-square perturbation is of order

$$\frac{\mathcal{P}_{\chi^2}(k, \tau)}{\chi^4(t)} \simeq 4 \left(\frac{k}{\mu} \right)^3 \tau^{3/4}. \quad (5.18)$$

This will be much less than 1 except perhaps on the very smallest scales $k \sim H$. As a result, δP_χ can be treated as a first-order cosmological perturbation.

§6. Primordial curvature perturbation ζ

6.1. Time-dependence of ζ

To define ζ one smooths the metric and the energy-momentum tensor on a comoving scale k_{smooth} , that is outside the horizon, yet below the shortest scale of interest.^{1) #4} In the gauge with the comoving threading and the slicing of uniform energy density, the spatial metric defines ζ through the expression

$$g_{ij}(\mathbf{x}, t) \equiv a^2(t) e^{2\zeta(\mathbf{x}, t)} \gamma_{ij}(\mathbf{x}, t). \quad (6.1)$$

In this expression, a is the unperturbed scale factor, and γ has unit determinant.

The local scale factor (such that a comoving volume is proportional to a^3) is $a(\mathbf{x}, t) \equiv a(t) \exp[\zeta(\mathbf{x}, t)]$. Since the smoothing scale is outside the horizon, no energy can flow on bigger scales which means that the energy continuity equation is satisfied locally:

$$\dot{\rho}(t) = -3 \frac{\dot{a}(\mathbf{x}, t)}{a(\mathbf{x}, t)} [\rho(t) + P(t) + \delta P_{\text{nad}}(\mathbf{x}, t)], \quad (6.2)$$

where δp_{nad} is the pressure perturbation on the slicing of uniform density (non-adiabatic pressure perturbation). We can choose the unperturbed quantities $P(t)$

^{#4} A function of position is said to be smooth on a given scale k_{smooth} , if it has no fourier components with bigger k . If a function has such modes, it is smoothed by removing them.

and $\rho(t)$ so that the unperturbed energy continuity equation (2.9) is satisfied. Then the time-dependence of ζ is given by

$$\dot{\zeta}(\mathbf{x}, t) = -\frac{H(t)}{\rho(t) + P(t) + \delta p_{\text{nad}}(\mathbf{x}, t)} \delta p_{\text{nad}}(\mathbf{x}, t), \quad (6.3)$$

where $H \equiv \dot{a}(t)/a(t)$. During any era when P is a unique function of ρ , δp_{nad} vanishes and ζ is constant.

To first order in the perturbations, we have

$$\dot{\zeta}_{\mathbf{k}}(t) = -\frac{H(t)}{\rho(t) + P(t)} \delta P_{\mathbf{k}}^{\text{nad}}(t), \quad (6.4)$$

and

$$\delta P_{\mathbf{k}}^{\text{nad}}(t) = \delta P_{\mathbf{k}}(t) - \frac{\dot{P}(t)}{\dot{\rho}(t)} \delta \rho_{\mathbf{k}}(t) = \delta P_{\mathbf{k}}(t) + \frac{\dot{P}(t)}{3H(\rho(t) + P(t))} \delta \rho_{\mathbf{k}}(t), \quad (6.5)$$

where the right hand side can be evaluated in any gauge.

6.2. The waterfall contribution to ζ

We define the waterfall contribution to ζ as

$$\zeta_{\chi\mathbf{k}} \equiv \zeta_{\mathbf{k}}(\tau_{\text{nl}}) - \zeta_{\mathbf{k}}(\tau_*) = -\frac{H}{\mu} \int_{\tau_*}^{\tau_{\text{nl}}} d\tau \frac{\delta P_{\mathbf{k}}^{\text{nad}}(\tau)}{\rho(t) + P(t)}, \quad (6.6)$$

with $1 \ll \tau_* \ll \tau_{\text{nl}}$. We are going to see that the integral is dominated by the upper limit for any choice τ_* in this range.

The inflaton contribution to δP_{nad} vanishes (corresponding to $\dot{\zeta}_\phi = 0$) and the waterfall contribution is just δP_χ . Also, the consistency condition (3.6) implies $\dot{\phi}^2 \gg \dot{\chi}^2$, which means that $\rho + P \simeq \dot{\phi}^2$. Using Eq. (5.11) we therefore have

$$\zeta_{\chi\mathbf{k}} \simeq -\frac{H\mu}{\dot{\phi}^2} \int_{\tau_*}^{\tau_{\text{nl}}} d\tau \tau [\delta(\chi^2(\tau))]_{\mathbf{k}}, \quad (6.7)$$

where $[\delta(\chi^2(\tau))]_{\mathbf{k}}$ is the Fourier transform of $\delta[\chi^2(t, \mathbf{x})]$. Since $\delta[\chi^2(\mathbf{x}, \tau)]$ increases like $\exp(\frac{4}{3}\tau^{3/2})$ up to a prefactor, we have in the limit $\tau_{\text{nl}} \rightarrow \infty$

$$\zeta_{\chi\mathbf{k}} = -\frac{1}{2} \frac{H\mu}{\dot{\phi}^2} \tau_{\text{nl}}^{1/2} [\delta(\chi^2(\tau))]_{\mathbf{k}}, \quad (6.8)$$

which gives

$$\mathcal{P}_{\zeta_\chi}(k) \simeq \frac{36}{\sqrt{2\pi}} \tau_{\text{nl}}^{7/4} \left(\frac{H}{\mu}\right)^7 \left(\frac{k}{H}\right)^3 = \frac{36}{\sqrt{2\pi}} \tau_{\text{nl}}^{-21/4} (H t_{\text{nl}})^7 \left(\frac{k}{H}\right)^3. \quad (6.9)$$

The black hole bound is $\mathcal{P}_{\zeta_\chi}(k=H) \lesssim 10^{-2}$. This will almost certainly be satisfied by Eq. (6.9), because it is derived under our assumptions that include $H t_{\text{nl}} \ll 1$ and $\tau_{\text{nl}} \gg 1$.

On a bigger scale $k = e^{-N}H$, leaving the horizon N Hubble times before the end of inflation, the k^3 dependence makes $\mathcal{P}_{\zeta_\chi} \lesssim 10^{-(2+1.3N)}$ negligible compared with the total observed value $\mathcal{P}_\zeta(k) \sim 10^{-9}$ unless $N(k) < 5$. This means that \mathcal{P}_{ζ_χ} is negligible on cosmological scales unless $N_0 < 20$, which is impossible with any reasonable post-inflationary cosmology even with a low inflation scale. We conclude that the black hole constraint on the scale $k = H$ makes \mathcal{P}_{ζ_χ} negligible on cosmological scales, with our assumptions.

§7. Conclusion

In this paper we have made some observations on the usual procedure of dropping the quantum regime. Keeping only the classical regime, we went on follow the waterfall field and its contribution to the pressure perturbation. Then we arrived at an expression for the contribution to the curvature perturbation that is generated during the waterfall. To do that, we made simplifying assumptions; that the waterfall starts during slow-roll inflation, that it takes much less than a Hubble time and that ϕ and $\dot{\phi}$ have negligible variation. On the other hand, we assumed nothing about the form of the inflationary potential. Our results depend on five parameters; the Hubble parameter H , the tachyonic mass-squared m^2 and self-coupling λ of the waterfall, its coupling g^2 to the inflaton, and fraction \sqrt{f} of the observed curvature perturbation that is generated by the inflaton perturbation. We evaluated, for the first time, the region of parameter space in which these assumptions hold.

The fundamental feature of our calculation (shared by the discussion in²⁵⁾ for preheating after ϕ^2 chaotic inflation) is that the *linear* non-adiabatic pressure perturbation is generated by terms that are *quadratic* in the field perturbation. We find that the contribution to the curvature perturbation, generated by the waterfall field during the waterfall, has a spectrum proportional to k^3 , making it negligible on cosmological scales. We concur with the general view that such a result will apply to any contribution generated by a field that is massive during inflation, though we have no universal proof for the waterfall let alone for the general case.

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